

TOWARD A NON-ABELIAN
ATIYAH-BREDON SEQUENCE

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Toward a non-Abelian Atiyah-Bredon Sequence

by

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Abstract

For a topological group G acting on a G -CW complex, the Chang-Skjelbred Lemma presents an exact sequence of equivariant cohomology groups [7]. Atiyah and Bredon determined an exact sequence that extends the Chang-Skjelbred Lemma [7]. The Atiyah-Bredon sequence holds only for torus actions, but Baird has conjectured that it can be generalized to actions by any compact connected Lie group. This work provides the necessary background in equivariant cohomology for this conjecture.

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0.1 Introduction

Consider a topological group G acting on a topological space X . We can associate an algebraic invariant to this action called its equivariant cohomology. This creates a graded ring $H_G^*(X)$ that is functorial in the sense that given a G -equivariant map $\phi : X \rightarrow Y$ between two topological spaces X and Y , there is a map $\phi^* : H_G^*(Y) \rightarrow H_G^*(X)$. In particular, considering the map $\psi : X \rightarrow *$ sending every point of X to a single point, the induced map $\psi^* : H_G^*(*) \rightarrow H_G^*(X)$ allows us to consider $H_G^*(X)$ as a module over $H_G^*(*)$.

Let $T = (\mathbb{S}^1)^r$ be a torus of dimension r and X a compact, smooth manifold. Suppose that X is *equivariantly formal*, by which we mean that its equivariant cohomology $H_T^*(X)$ is a free module over the polynomial ring $H_T^*(*)$. Through the work of Atiyah and Bredon [7], we know that the sequence

$$0 \rightarrow H_T^*(X) \xrightarrow{i^*} H_T^*(X_0) \xrightarrow{\delta_0} H_T^{*+1}(X_1, X_0) \xrightarrow{\delta_1} H_T^{*+2}(X_2, X_1) \xrightarrow{\delta_2} \\ H_T^{*+3}(X_3, X_2) \xrightarrow{\delta_3} \dots \xrightarrow{\delta_r} H_T^{*+r}(X_r, X_{r-1}) \rightarrow 0$$

is exact, where δ_i is the transition map from the long exact sequence of the triple (X_{i+1}, X_i, X_{i-1}) , and X_i denotes the union of all orbits of dimension at most i .

The Atiyah-Bredon sequence holds only for torus actions, but Baird has conjectured that it can be generalized to actions by any compact connected Lie group. Let

G be a compact, connected, rank r Lie group, and let $X_i = \{p \in X : G_p^0 \text{ has rank } \geq r - i\}$, where G_p^0 is the identity component of the stabilizer of p . The conjecture states that the sequence

$$0 \rightarrow H_G^*(X) \xrightarrow{i^*} H_G^*(X_0) \xrightarrow{\delta_0} H_G^{*+1}(X_1, X_0) \xrightarrow{\delta_1} H_G^{*+2}(X_2, X_1) \xrightarrow{\delta_2} \\ H_G^{*+3}(X_3, X_2) \xrightarrow{\delta_3} \dots \xrightarrow{\delta_r} H_G^{*+r}(X_r, X_{r-1}) \rightarrow 0,$$

known as the non-abelian Atiyah-Bredon sequence, is exact. Exactness of the first four terms has already been established, a result known as the non-abelian Chang-Skjelbred Lemma [9].

The purpose of this thesis is to provide the necessary background in equivariant cohomology for this conjecture. The main prerequisites are a familiarity with smooth manifolds and Lie groups, for which the author recommends Lee's *Introduction to Smooth Manifolds* [12]; and ordinary cohomology theory, for which Hatcher's *Algebraic Topology* [10] is the standard text.

Chapter 1

General Mathematical

Preliminaries

1.1 Compact Tori

Recall that the unit circle \mathbb{S}^1 , considered as a subset of the complex plane, is a Lie group with complex multiplication as the group operation. The Cartesian product of n circles $\mathbb{T}^n = \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ is known as an n -torus—so named because the familiar torus from geometry is diffeomorphic to the 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, where \mathbb{T}^2 has the product smooth manifold structure. \mathbb{T}^n is an abelian Lie group with the group structure given by componentwise multiplication. We also write $\mathbb{T}^r = (\mathbb{S}^1)^r$, and call

r the *rank* of the torus.

Let G be a compact connected Lie group. A *maximal torus* of G is a Lie subgroup $T \subseteq G$, where T is isomorphic to a torus and not properly contained in any abelian Lie subgroup of G . Since tori are abelian Lie groups, this means in particular that a maximal torus of G is not contained in any strictly larger Lie subgroup of G isomorphic to a torus.

Theorem 1.1.1 (Existence of Maximal Tori). *Let G be a compact connected Lie group. Then G contains a maximal torus, and every element of G is contained in a maximal torus of G . Furthermore, all the maximal tori of G are conjugate. This means that for any two maximal tori H and K , there exists an element $g \in G$ such that $K = gHg^{-1}$.*

Proof. See [6]. □

Theorem 1.1.2 (Closed Subgroup Theorem). *Suppose G is a Lie Group, and $H \subset G$ is a subgroup that is also a closed subset. Then H is an embedded Lie subgroup.*

Proof. This is Theorem 15.29 of [12]. □

Example 1.1.1. Recall that a complex valued matrix A is unitary if $A^*A = I_n$, where A^* is the conjugate transpose of A . The group of all n -dimensional unitary

matrices $U(n)$ forms a compact, connected Lie group and therefore, by Theorem 1.1.1, contains a maximal torus.

Let $D(n)$ be the set of $n \times n$ diagonal unitary matrices. $D(n)$ is a group because it contains I_n , and is closed under inverses and matrix multiplication. Now, any matrix $A \in U(n) \setminus D(n)$ has a nonzero entry a_{ij} where $i \neq j$. Identifying $U(n)$ with a subspace of \mathbb{C}^{n^2} , we see that A is contained in an open ball of radius $|a_{ij}|$ centred at A . Thus $U(n) \setminus D(n)$ is an open subset of $U(n)$, whereby $D(n)$ is closed. It follows from 1.1.2 that $D(n)$ is a Lie subgroup of $U(n)$.

Let $A \in D(n)$. Then the i th diagonal entry of A^*A is equal to $\overline{A_{ii}}A_{ii}$. As $A^*A = I_n$, we have that $\overline{A_{ii}}A_{ii} = 1$. This is equivalent to the condition that A_{ii} lie on the unit circle. From this we see that $D(n)$ is isomorphic to \mathbb{T}^n through the map taking each circle to a diagonal entry of $D(n)$. To show that $D(n)$ is a maximal torus it remains to show that $D(n)$ is not properly contained in any abelian Lie subgroup of $U(n)$.

Suppose that G is an abelian Lie subgroup of $U(n)$ which contains $D(n)$ as a proper subgroup. Since $D(n)$ is a proper subgroup of G , G must contain a matrix A with a nonzero non-diagonal entry A_{jk} . Let B be the diagonal matrix with $B_{jj} = i$ and $B_{kk} = 1$, for $j \neq k$. Then $(BA)_{jk} = B_{jj}A_{jk} = iA_{jk}$, but $(AB)_{jk} = A_{jk}B_{kk} = A_{jk}$. This contradicts our assumption that G is an abelian group, so $D(n)$ cannot be a proper subgroup of any abelian subgroup of $U(n)$. Therefore, $D(n)$ is a maximal

torus.

1.2 Topological Group Actions

A *topological group* is a topological space with a group structure such that the multiplication map $m : G \times G \rightarrow G$ and inversion map $i : G \rightarrow G$, given by

$$m(g, h) = gh, \quad i(g) = g^{-1},$$

are both continuous.

For example, the torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$ is a topological group with multiplication map $m((w_1, w_2), (z_1, z_2)) = (w_1 z_1, w_2 z_2)$. The general linear group $GL_n(\mathbb{C})$ of invertible $n \times n$ matrices becomes a topological group when considered as a subspace of \mathbb{C}^{n^2} .

Given a set S , a *left action* of G on X is a map $G \times X \rightarrow X$, often written as $(g, p) \mapsto g \cdot p$, that satisfies

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \text{for all } g_1, g_2 \in G \text{ and } x \in X;$$

$$e \cdot x = x \quad \text{for all } x \in X.$$

A *right action* is defined analogously.

Now suppose that G is a topological group and X is a topological space. An

action of G on X is said to be *continuous* if the map $G \times X \rightarrow X$ defining the action is continuous. In this case X is said to be a G -space.

For a point $p \in X$, the *stabilizer* of p is the set G_p of all elements $g \in G$ for which $g \cdot p = p$. It is not hard to show that G_p is a closed subgroup of G . Consequently, if G is a Lie group, then G_p is a Lie subgroup of G by Theorem 1.1.2. An action is said to be *free* if the stabilizer of every point is the trivial subgroup.

Let X be a G -space with left action ρ . For each $g \in G$, since ρ is continuous it restricts to a homeomorphism $\rho : \{g\} \times X \rightarrow X$. In this way each element of G corresponds to a homeomorphism of X onto itself. If each element of G induces a different homeomorphism, then we call ρ an *effective action*. When G acts effectively on X , G corresponds to a group of homeomorphisms of X onto itself.

Example 1.2.1. Let $G = \{z \in \mathbb{C} : |z| = 1\}$ be the circle group and $X = \mathbb{S}^2$ be the 2-sphere. Let $p = (\theta, \phi)$ be a point on X in spherical coordinates (θ representing the azimuthal angle and ϕ the polar angle). Define an action of G on X by $z \cdot p = (z\theta, \phi)$. This action has the effect of rotating X about its polar axis. It is not a free action, because the two poles have stabilizer groups equal to G . It is effective, however, because each $z \in G$ corresponds to a unique rotation. By using a suitable coordinate transformation, we could use the same action of G on \mathbb{S}^2 to rotate it about any other line through the origin.

Suppose G is a topological group, and X and Y are left G -spaces. An *equivariant map* is a continuous function $f : X \rightarrow Y$ that respects the group actions on X and Y . This means that for all $x \in X$ and $g \in G$,

$$f(g \cdot x) = g \cdot f(x).$$

Intuitively, this says that we receive the same result whether we allow G to act before or after evaluating the function f . We also call f a G -map.

For a given topological group G , we can form a category by taking the collection of G -spaces as objects and the collection of G -maps as morphisms. Equivariant maps are convenient tools in the study of group actions because they allow us to express the properties of an action on one space in terms of an action on another space that may be better understood.

1.2.1 Principal Bundles

Let G be a topological group. Suppose that we have a G -space E with the property that for every point x of E there is a topological space U_x , an open neighbourhood V_x of x , and a homeomorphism $\phi_x : V_x \rightarrow U_x \times G$. Define an action of G on $U_x \times G$ by $h \cdot (x, g) = (x, hg)$. We say that E is a *principal G -bundle* if, for each x in E , V_x is invariant under the action of G on E , and ϕ_x is an equivariant map with respect

to the actions on E and $U_x \times G$. We say that two principal G -bundles X and Y are *isomorphic* if there is a homeomorphism $h : X \rightarrow Y$ such that h is a G -equivariant map.

Example 1.2.2. Let G be the additive group $\mathbb{Z}_2 = \{0, 1\}$ and X be the circle \mathbb{S}^1 . Define an action of G on X by $0 \cdot z = z$, but $1 \cdot z = ze^{i\pi}$. Thus the non-zero element of G takes a point on X to its antipodal point. For $x = e^{i\theta} \in X$, let U_x be the arc $\{e^{i\phi} : \theta - \pi/4 < \phi < \theta + \pi/4\}$ centred on θ , and $U'_x = \{ze^{i\pi} : z \in U_x\}$ be the same arc on the opposite side of the circle. Let $V_x = U_x \cup U'_x$. Then V_x is an open neighbourhood of x , and the map $\phi_x : V_x \rightarrow U_x \times G$ given by

$$\phi_x(z) = \begin{cases} (z, 0) & \text{if } z \in U_x \\ (ze^{i\pi}, 1) & \text{if } z \in U'_x \end{cases}$$

is a homeomorphism. If $z \in U_x$, then $\phi_x(0 \cdot z) = \phi_x(z) = (z, 0) = 0 \cdot \phi_x(z)$, and $\phi_x(1 \cdot z) = \phi_x(ze^{i\pi}) = ([ze^{i\pi}]ze^{i\pi}, 1) = 1 \cdot (z, 0) = 1 \cdot \phi_x(z)$. The reader can verify that the identity $\phi_x(g \cdot z) = g \cdot \phi_x(z)$ holds equally for $z \in U'_x$, whereby ϕ_x is an equivariant map. This shows that the circle \mathbb{S}^1 is a \mathbb{Z}_2 -principal bundle. In fact, \mathbb{S}^n can be given the structure of a \mathbb{Z}_2 -principal bundle for any n : Instead of the arcs used above we consider the analogous n -dimensional regions.

The next theorem is valid for all “completely regular” G -spaces. We do not define this term, since we are not working in such generality. It can be shown that all

metrizable spaces are completely regular, which naturally includes compact smooth manifolds, our chief object of study.

Theorem 1.2.1. *Let G be a compact Lie group acting freely on a completely regular space X . Then X is a principal G -bundle.*

Proof. Follows from Theorem 3.6 in [8]. □

1.3 Fibre Bundles

It is a routine practice in mathematics to take two topological spaces X and Y , and create a new topological space $X \times Y$ by endowing their Cartesian product with the product topology. A cylinder can be modelled as the product $\mathbb{S}^1 \times \mathbb{I}$ of a circle with an interval, and the torus as the product $\mathbb{S}^1 \times \mathbb{S}^1$ of two circles. Some spaces cannot be described this way, but nonetheless look locally like a Cartesian product. For example, every point of the Möbius band is contained in a region homeomorphic to a rectangle, but the twisted topology of the Möbius band prevents us from expressing it globally as a Cartesian product. It is this example that motivates our next definition.

A *fibre bundle* is a collection $\{E, B, \pi, F\}$ with the following properties:

1. F , E , and B are topological spaces, called the *fibre*, *total space*, and *base space* of the bundle respectively.
-

2. $\pi : E \rightarrow B$ is a continuous surjection, called the *projection map*.
3. For each point $x \in E$, there is an open subset $U_i \subseteq B$ such that $\pi(x) \in U_i$, and a homeomorphism $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$.
4. For each $(x, a) \in U_i \times F$, $\pi \circ \phi_i^{-1}(x, a) = x$.

The last condition can be expressed by saying that the following diagram commutes,

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\phi} & U_i \times F \\ \downarrow \pi & \swarrow \pi_1 & \\ U_i & & \end{array}$$

where π_1 represents projection onto the first factor.

A fibre bundle is thus a space that looks locally like a Cartesian product, but may have different global topological properties. We express this by saying that E is *locally trivial*, and call the collection $\{(U_i, \phi_i)\}$ the *local trivializations* of E . The spaces U_i are called *trivializing neighbourhoods*. If E is just the Cartesian product $U \times F$ of a topological space U with a fibre F , then in the above definition we take each U_i to be U and call E a *trivial product*.

We generally visualize the total space of a fibre bundle as a collection of fibres suspended above the base space, each fibre being affixed to the base space at a unique point determined by the projection map.

Example 1.3.1. Let $\mathbb{I} = [0, 1]$ be the closed unit interval, and $E = (\mathbb{I} \times \mathbb{R}) / \sim$ be an infinite strip with its outer edges identified through the relation $(0, y) \sim (1, -y)$. With this construction E is a model of the Möbius band. It is also the total space of a fibre bundle $\{E, B, \pi, F\}$, with $B = \mathbb{S}^1$, $F = \mathbb{R}$, and $\pi(x, y) = e^{2\pi i x}$, the base space representing the central circle of the band.

Example 1.3.2. The reader will be familiar with the concept of a vector bundle. In this context a rank k vector bundle is simply a fibre bundle whose fibre is a k -dimensional vector space, and where the locally trivializing maps ϕ_i restrict to linear isomorphisms $\pi^{-1}(x) \rightarrow \{x\} \times F$.

Let E be a principal G -bundle, as defined in Section 1.2.1. We can take the collection $\{(U_i, \phi_i)\}$ as the trivializing neighbourhoods of a fibre bundle with base space E/G , where E/G is the set of orbits of points in E . The projection map π is defined by sending an element of E to its orbit in E/G , and E/G is given the quotient topology with quotient map π . Abusing our notation slightly, we also call the resulting fibre bundle $\mathcal{P} = \{E, E/G, \pi, G\}$ a principal G -bundle.

Let $\mathcal{B} = \{E, B, \pi, F\}$ be a fibre bundle, B' be a topological space, and $f : B' \rightarrow B$ be a continuous map. We can use f to define a fibre bundle over B' with the same fibre as \mathcal{B} . Let $f^*E = \{(x, e) \in B' \times E : f(x) = \pi(e)\}$, and give f^*E the subspace topology. Define a projection map $\pi' : f^*E \rightarrow B'$ by $\pi'(x, e) = x$. Then $\{f^*E, B', \pi', F\}$ is a

fibre bundle, called the *pullback bundle*, which we denote $f^*\mathcal{B}$.

1.3.1 Constructing a Fibre Bundle

Given a fibre bundle $\mathcal{B} = \{E, B, \pi, F\}$, let (U_i, ϕ_i) and (U_j, ϕ_j) be two overlapping trivializing neighbourhoods. For any $x \in U_i \cap U_j$, by definition the maps ϕ_i and ϕ_j each restrict to homeomorphisms from $\pi^{-1}(x)$ to $\{x\} \times F$. Thus the map $\phi_i \phi_j^{-1}$ restricts to a homeomorphism from $\{x\} \times F$ to itself. If we let G be a group of homeomorphisms of F onto itself, then for each $x \in B$, the map $\phi_i \phi_j^{-1}$ can be identified with an element of G . We thus define a map $\phi_i \phi_j^{-1} : (U_i \cap U_j) \times F \rightarrow (U_i \cap U_j) \times F$ by

$$\phi_i \phi_j^{-1}(x, \xi) = (x, t_{ij}(x)\xi)$$

where for each $x \in U_i \cap U_j$, the *transition function* $t_{ij} : U_i \cap U_j \rightarrow G$ evaluates to the homeomorphism of F onto itself described above. G is said to be the *structure group* of \mathcal{B} .

Note that the definition of a transition function given above is not dependent on the trivializing neighbourhoods it is derived from. We can construct many examples of fibre bundles by starting with a base space, a fibre, and a collection of functions having the properties of transition functions.

Theorem 1.3.1 (Existence of Fibre Bundle). *Let G be a topological group, B a*

topological space, and F a G -space. Suppose that we have an open covering $\{U_i\}_{i \in I}$ of B and, for each $i, j \in I$, a continuous map

$$t_{ji} : U_i \cap U_j \rightarrow G$$

such that

$$t_{kj}(x)t_{ji}(x) = t_{ki}(x),$$

whenever $x \in U_i \cap U_j \cap U_k$, and such that $t_{ii}(x)$ is the identity element of G for all $i \in I$ and $x \in U_i$. Then there exists a unique fibre bundle with base space B , fibre F , and transition functions $\{t_{ij}\}$.

Proof. See [14] for the full proof; we give the construction here. Let

$$E = \left(\bigsqcup_{i \in I} U_i \times F \right) / \sim,$$

the disjoint union of all sets of the form $U_i \times F$ under the equivalence relation $(u, x) \sim (v, y)$ when $u = v$ and $t_{ji}(u) \cdot x = y$, where $(u, x) \in U_i \times F$ and $(v, y) \in U_j \times F$. The projection map $\pi : E \rightarrow B$ is then given by $\pi(u, x) = u$, and this is well-defined because all elements identified under \sim have equal first terms. \square

Example 1.3.3. We can use this method to construct the Möbius band. Let B be the unit interval with the ends identified, let $F = \mathbb{R}$, and let G be the additive group $\mathbb{Z}_2 = \{0, 1\}$. The non-zero element of G acts on F through multiplication by -1 . Let

$U_1 = (\frac{1}{5}, \frac{4}{5})$, and $U_2 = [0, \frac{2}{5}) \cup (\frac{3}{5}, 1]$ (Note that U_2 is a connected open subset of B because $0 \sim 1$). U_1 and U_2 form an open cover of B , and we obtain the Möbius band by applying Theorem 1.3.1 with the following transition maps:

$$t_{11}(x) = 0, \quad \forall x \in U_1,$$

$$t_{22}(x) = 0, \quad \forall x \in U_2,$$

$$t_{12}(x) = t_{21}(x) = 1, \quad \forall x \in (\frac{1}{5}, \frac{2}{5}),$$

$$t_{12}(x) = t_{21}(x) = 0, \quad \forall x \in (\frac{3}{5}, \frac{4}{5}).$$

Example 1.3.4. Let G be a topological group, and B a topological space. Suppose further that we have an open cover $\{U_i\}_{i \in I}$ of B and a set of transition functions satisfying the hypotheses of Theorem 1.3.1. If we allow G to act on itself by left translation, applying Theorem 1.3.1 we obtain a fibre bundle \mathcal{P} with fibre G . For each $i \in I$, it is clear from the above construction that $\pi^{-1}(U_i) \cong U_i \times G$, and the reader can verify that this homeomorphism is an equivariant map. Hence \mathcal{P} is a principal G -bundle.

1.3.2 Associated Bundles

Let $\mathcal{P} = \{E, B, \pi, G\}$ be a principal bundle. Given a G -space X , let G act diagonally on $E \times X$. Thus, for $g \in G$ and $(e, x) \in E \times X$, we have $g \cdot (e, x) = (g \cdot e, g \cdot x)$, using

the actions of G on E and X respectively. We will denote by $E \times_G X$ the quotient of $E \times X$ by this action.

Consider the projection map $\pi : E \times X \rightarrow E$. As π is equivariant, it determines a map $\Pi : E \times_G X \rightarrow E/G$ by sending $[(p, x)]$ to $[\pi(p, x)] = [p]$. With this map, the bundle $\{E \times_G X, B, \Pi, X\}$ is a fibre bundle, known as the X -associated bundle to \mathcal{P} .

Example 1.3.5. Let G be a topological group acting effectively on a topological space X , and $\{V_j\}, \{t_{ij}\}$ be a system of coordinate transformations in a topological space B . Theorem 1.3.1 asserts the existence of an unique fibre bundle \mathcal{F} with base space B and fibre X . We could equally, however, allow G to act on itself by left translation, as we did in Example 1.3.4, then apply Theorem 1.3.1 with the same coordinate transformations to obtain a principal bundle \mathcal{P} . It can be shown that if we form the X -associated bundle to \mathcal{P} , the resulting fibre bundle is isomorphic to \mathcal{F} .

1.3.3 Classifying Spaces

The proofs for this section can be found in [13].

Theorem 1.3.2. *Let B be a topological space, and P a principal G -bundle over B . Let X be a CW-complex and $f, g : X \rightarrow B$ be homotopic maps. Then the pullback bundles f^*P and g^*P are isomorphic as principal G -bundles over X .*

We will use the notation $[X, B]$ to represent the set of equivalence classes of homotopic maps from X to B . A connected topological space X is said to be *weakly contractible* if its homotopy group $\pi_n(X)$ of each dimension is trivial.

Theorem 1.3.3. *Suppose that $P \rightarrow B$ is a principal G -bundle with P weakly contractible. Then for any CW-complex X , the map taking f to f^*P is a bijection from $[X, B]$ to the set of isomorphism classes of principal G -bundles over X .*

If P is a weakly contractible principal G -bundle, then its base space is referred to as a *classifying space* for G , and P is called a universal G -bundle.

Theorem 1.3.4. *Every topological group G has a CW-classifying space (A CW-classifying space is a classifying space that is also a CW-complex). Any two CW-classifying spaces for G are homotopy equivalent.*

As a consequence of Theorem 1.3.4, we generally denote “the” classifying space of a topological group as BG , and “the” universal bundle over BG as EG , bearing in mind that these spaces are unique only up to homotopy equivalence.

Example 1.3.6. The classifying space for \mathbb{S}^1 is $\mathbb{C}P^\infty$, and the universal bundle $E\mathbb{S}^1$ is \mathbb{S}^∞ .

Example 1.3.7. If we have two topological groups G and H , the classifying space $B(G \times H)$ of their Cartesian product is simply $BG \times BH$, and similarly $E(G \times H) =$

$EG \times EH$. Thus, by Example 1.3.6, if T is a torus of rank r , then $BT = (\mathbb{C}P^\infty)^r$ and $ET = (\mathbb{S}^\infty)^r$.

Example 1.3.8. The *Grassmannian* $Gr(n, V)$ is the set of all n -dimensional subspaces of the vector space V . An *orthonormal n -frame* is a set of n linearly independent orthonormal vectors. We denote the set of orthonormal n -frames over a vector space V by $Fr(n, V)$. It can be shown that the classifying space for $U(n)$ is the Grassmanian $Gr(n, \mathbb{C}^\infty)$, and that $EU(n) = Fr(n, \mathbb{C}^\infty)$. $U(n)$ acts on the frames as follows: If $A = (A_{ij}) \in U(n)$ and $v = (v_1, \dots, v_n)$ is an orthonormal n -frame, then

$$A \cdot v = (A_{11}v_1 + \dots + A_{1n}v_n, A_{21}v_1 + \dots + A_{2n}v_n, \dots, A_{n1}v_1 + \dots + A_{nn}v_n).$$

Example 1.3.9. If H is a subgroup of G , then $EH = EG$, and $BH = EG/H$. Consider \mathbb{Z}_2 as a subgroup of \mathbb{S}^1 , referring back to Example 1.3.6. As $ES^1/\mathbb{Z}_2 = \mathbb{S}^\infty/\mathbb{Z}_2 = \mathbb{R}P^\infty$, we obtain a principal \mathbb{Z}_2 -bundle $\mathbb{S}^\infty \rightarrow \mathbb{R}P^\infty$.

Chapter 2

Equivariant Cohomology Presented

2.1 Introduction

If we are given a fibre bundle $E \rightarrow B$, we can take advantage of a large amount of existing theory relating the cohomology groups of the total space E to those of the base space B . Suppose, instead, that we have a G -space X , and we want to work out the cohomology groups of X . If G acts freely on X , then Theorem 1.2.1 tells us that X is the total space of a principal G -bundle. The base space of this bundle may be better understood, and we can relate its cohomology groups to those of X . In the case where the action is not free, we can look for a space homotopic to X with a free action by G . If G is a topological group, then Theorem 1.3.4 asserts the existence of a principal G -

bundle EG . Since EG is weakly contractible, there is a homotopy equivalence between the product $EG \times X$ and X , whereby they have the same cohomology groups. G acts freely on $EG \times X$, and the associated bundle construction allows us to concretely determine the base space of the corresponding principal bundle. The cohomology of this base space and its relation to the cohomology of X are the primary objects of investigation in *equivariant cohomology*, this name deriving from the equivariant map used to construct the associated bundle.

2.2 Borel Construction

To simplify the theory we only consider cohomology with complex coefficients, though the results in this paper are valid with coefficients taken from any field of characteristic 0. We will use the asterisk $*$ to denote a topological space containing a single point.

The G -spaces we consider will be finite G -CW-complexes, for the details of which the reader is referred to the first chapter of [1]. However, in many applications of equivariant cohomology we are only concerned with compact smooth manifolds acted upon by compact Lie groups, and such manifolds can be realized as G -CW-complexes.

Let G be a topological group, and X a G -space. Take the universal G -bundle EG and form the X -associated bundle $\{EG \times_G X, BG, \pi, G\}$. The *equivariant cohomology*

of X , denoted $H_G^*(X)$, is the ordinary cohomology $H^*(EG \times_G X)$ of the associated bundle.

For an arbitrary G -space X , it is in general a difficult problem to study the cohomology of the orbit space X/G . One of the motivations for studying the equivariant cohomology of G -spaces is to better understand these quotient spaces. Since $X \sim EG \times X$ (EG is contractible), we can often learn a lot about X/G by studying $EG \times_G X$. In particular, if X happens to be a principal G -bundle, then the projection map

$$EG \times_G X \rightarrow X/G$$

is a homotopy equivalence, and thus

$$H_G^*(X) \cong H^*(X/G).$$

This can be also seen as a consequence of Theorem 2.3.1 below.

Example 2.2.1. Let G be a topological group acting freely on itself, and consider the G -associated bundle $EG \times_G G$. For $(e, g) \in EG \times G$, we have $g^{-1} \cdot (e, g) = (g^{-1}e, 1)$, so each equivalence class in $EG \times_G G$ has a representative of the form $(e, 1)$, for some $e \in EG$. Now suppose that $[(e, 1)] = [(f, 1)]$ for some $e, f \in EG$. In this case there exists a $g \in G$ such that $g \cdot (e, 1) = (f, 1)$, whereby $g \cdot 1 = 1$. Therefore $e = f$, and we have established a one-to-one correspondence between EG and $EG \times_G G$. In fact

the map $e \mapsto [(e, 1)]$ is a homeomorphism.

Since $EG \cong EG \times_G G$, and EG is contractible, $H_G^*(G) \cong H^*(EG) = H^*(*)$.

Hence the equivariant cohomology of a group acting freely on itself is the ordinary cohomology of a point.

2.3 General Properties

Suppose that a topological group G is acting trivially on a point $*$. Since $EG \times * \cong EG$, $H_G^*(*) \cong H^*(EG/G) = H^*(BG)$. The construction of equivariant cohomology involved forming a fibre bundle with projection map $\pi : EG \times_G X \rightarrow BG$. This map induces a map on cohomology rings $\pi^* : H^*(BG) \rightarrow H^*(EG \times_G X)$, or equivalently $\pi^* : H_G^*(*) \rightarrow H_G^*(X)$. We express the fact that there is a ring homomorphism from $H_G^*(*)$ into $H_G^*(X)$ by saying that $H_G^*(X)$ is an *algebra* over $H_G^*(*)$. Since $H_G^*(X)$ is graded as a ring, it is also graded as an $H_G^*(*)$ -algebra. If we have a G -map f between two G -spaces X and Y , the induced map f^* on cohomology is then an $H_G^*(*)$ -algebra homomorphism between the equivariant cohomology groups $H_G^*(X)$ and $H_G^*(Y)$. It follows that equivariant cohomology can be thought of as a contravariant functor from the category of G -spaces and G -maps to the category of graded $H_G^*(*)$ -algebras and $H_G^*(*)$ -algebra homomorphisms, which we shall denote by $H_G^*(-)$.

Example 2.3.1. Let $G = X = \mathbb{S}^1$, and have G act on itself by rotation. Since this action is free, we know from Example 2.2.1 that $H_G^*(X) = H^*(*) = \mathbb{C}$. By Example 1.3.6, $BG = \mathbb{C}P^\infty$. We know from basic cohomology theory that $H^*(\mathbb{C}P^\infty)$ is equal to \mathbb{C} in even degree, and trivial in odd degree. It is intuitive from this result that $H^*(\mathbb{C}P^\infty) \cong \mathbb{C}[u]$, the graded polynomial ring over u , where u is considered to have degree 2; a fact which can be demonstrated through the use of spectral sequences.

Since $H_G^*(X) = \mathbb{C}$ and $H^*(BG) = \mathbb{C}[u]$, the projection $\pi : EG \times_G X \rightarrow BG$ induces the ring homomorphism $\pi^* : \mathbb{C}[u] \rightarrow \mathbb{C}$. There is only one such homomorphism that preserves degree, so the map π^* must be given by $\pi^*(a + bu + cu^2 + du^3 + \cdots) = a$.

Many of the properties of ordinary cohomology have direct analogs in equivariant cohomology that can be established without too much effort. For instance, if X and A are G -spaces with $A \subseteq X$, we can form the relative cohomology ring $H_G^*(X, A) = H^*(EG \times_G X, EG \times_G A)$, and there exists a long exact sequence

$$\cdots \rightarrow H_G^{*-1}(A) \xrightarrow{\delta} H_G^*(X, A) \rightarrow H_G^*(X) \rightarrow H_G^*(A) \xrightarrow{\delta} H_G^{*+1}(X, A) \rightarrow \cdots$$

Similarly, if $B \subseteq A \subseteq X$, then there exists a long exact sequence

$$\cdots \rightarrow H_G^{*-1}(A, B) \xrightarrow{\delta} H_G^*(X, A) \rightarrow H_G^*(X, B) \rightarrow H_G^*(A, B) \xrightarrow{\delta} H_G^{*+1}(X, A) \rightarrow \cdots$$

It is also worth remarking that for path-connected groups G the zero-dimensional cohomology group of a space X is the same as its zero-dimensional equivariant cohomology group. This is because $H^0(X)$ can be identified with the set of locally constant maps from X to \mathbb{C} . Since X and $EG \times_G X$ have the same number of path components, these sets are the same.

Theorem 2.3.1. *Let G be a topological group, and X a G -CW complex. If X_0 is a subspace of X preserved by the action of G , and G acts freely on the complement $X \setminus X_0$, then there is a ring isomorphism $H_G^*(X/G, X_0/G) \cong H^*(X/G, X_0/G)$.*

Proof. Lemma A.1.4 of [5]. □

2.4 Localization

Let T denote a torus of any rank. If X is a T -space, then the set of fixed points of the T -action on X is denoted by X^T . Letting T act trivially on X^T , the inclusion map $i : X^T \hookrightarrow X$ is equivariant. We can therefore apply $H_G^*(-)$ to i to obtain an $H_T^*(*)$ -algebra homomorphism $i^* : H_T^*(X) \rightarrow H_T^*(X^T)$.

Since the action of T on X^T is trivial, $ET \times_T X^T = BT \times X^T$. Thus $H_T^*(X^T) = H^*(BT \times X^T) = H^*(BT) \otimes_{\mathbb{C}} H^*(X^T)$; the last equality is due to the Künneth formula.

We can show that $H^*(BT)$ is isomorphic to $\mathbb{C}[u_1, u_2, \dots, u_r]$, the polynomial ring

in r variables over \mathbb{C} , where r is the rank of T . From Example 1.3.7, $BT = (\mathbb{C}P^\infty)^r$.

We stated in Example 2.3.1 that $H^*(\mathbb{C}P^\infty) \cong \mathbb{C}[u]$. By Künneth's formula,

$$\begin{aligned} H^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) &= H^*(\mathbb{C}P^\infty) \otimes \cdots \otimes H^*(\mathbb{C}P^\infty) \\ &= \mathbb{C}[u] \otimes \cdots \otimes \mathbb{C}[u] \\ &= \mathbb{C}[u_1, u_2, \dots, u_r]. \end{aligned}$$

As a consequence, $H_T^*(X^T) \cong \mathbb{C}[u_1, \dots, u_r] \otimes_{\mathbb{C}} H^*(X^T)$.

Since $\mathbb{C}[u_1, \dots, u_r]$ is an integral domain, we can form its quotient field,

$$\mathbb{C}(u_1, \dots, u_r) = \left\{ \frac{p}{q} : p, q \in \mathbb{C}[u_1, \dots, u_r] \text{ and } q \neq 0 \right\}.$$

Since $H^*(BT) \otimes_{\mathbb{C}} H^*(X^T)$ is an algebra over $H^*(BT) = \mathbb{C}[u_1, u_2, \dots, u_r]$, we can extend it by scalars to obtain an algebra over $\mathbb{C}(u_1, \dots, u_r)$:

$$\begin{aligned} &\mathbb{C}(u_1, \dots, u_r) \otimes_{\mathbb{C}[u_1, u_2, \dots, u_r]} H_T^*(X^T) \\ &= \mathbb{C}(u_1, \dots, u_r) \otimes_{\mathbb{C}[u_1, u_2, \dots, u_r]} (H^*(BT) \otimes_{\mathbb{C}} H^*(X^T)) \\ &= (\mathbb{C}(u_1, \dots, u_r) \otimes_{\mathbb{C}[u_1, u_2, \dots, u_r]} \mathbb{C}[u_1, u_2, \dots, u_r]) \otimes_{\mathbb{C}} H^*(X^T) \\ &= \mathbb{C}(u_1, \dots, u_r) \otimes_{\mathbb{C}} H^*(X^T). \end{aligned}$$

Theorem 2.4.1 (Localization). *Let X be a finite G -CW-complex. If we extend both $H_T^*(X)$ and $H_T^*(X^T)$ by scalars in $\mathbb{C}(u_1, \dots, u_r)$, then the induced inclusion map $i^* : H_T^*(X) \rightarrow H_T^*(X^T)$ becomes an isomorphism of rings $i^* : \mathbb{C}(u_1, \dots, u_r) \otimes_{\mathbb{C}[u_1, u_2, \dots, u_r]} H_T^*(X) \rightarrow \mathbb{C}(u_1, \dots, u_r) \otimes_{\mathbb{C}} H^*(X^T)$.*

Proof. Presented in a stronger form as Theorem (3.5) in [2]. □

2.5 Equivariantly formal G -spaces

A G -space X is called *equivariantly formal* if $H_G^*(X)$ is a free module over $H^*(BG)$.

Theorem 2.5.1. *Let X be a G -space, where G is a compact, connected Lie group. Then X is equivariantly formal if and only if there is an isomorphism of graded $H^*(BG)$ -modules from $H^*(X) \otimes_{\mathbb{C}} H^*(BG)$ to $H_G^*(X)$.*

Proof. See [3]. □

Example 2.5.1. Suppose T acts trivially on X . As T fixes every point, $X^T = X$, and thus $H_T^*(X) = H_T^*(X^T) = H^*(BT) \otimes_{\mathbb{C}} H^*(X^T)$. This is an isomorphism of graded $H^*(BT)$ -modules, so X is an equivariantly formal T -space.

Example 2.5.2. For readers familiar with Hamiltonian actions, consider the case where G is a compact, connected Lie group, X is a compact symplectic manifold, and a moment map $\mu : X \rightarrow \mathfrak{g}^*$ exists. Proposition 5.8 of [11] asserts that $H_G^*(X) \cong H^*(X) \otimes_{\mathbb{C}} H^*(BG)$, whereby X is equivariantly formal.

Theorem 2.5.2 (Lacunary Principle). *Let X be a finite CW-complex such that $H^k(X; \mathbb{C}) = 0$ for all odd k . Then any compact, connected, Lie group action on X is equivariantly formal.*

Proof. See [4]. □

Example 2.5.3. Consider $T = \mathbb{S}^1$ acting on $X = \mathbb{S}^2$ by rotation, as in Example 1.2.1.

By Theorem 2.5.2, this action is equivariantly formal, and so by the Localization Theorem

$$\begin{aligned} H_T^*(\mathbb{S}^2) &= H^*(\mathbb{S}^2) \otimes_{\mathbb{C}} H^*(BT) \\ &= H^*(\mathbb{S}^2) \otimes_{\mathbb{C}} \mathbb{C}[u] \\ &= \mathbb{C}[u] \cdot 1 \oplus \mathbb{C}[u] \cdot \alpha, \end{aligned}$$

where α is an element of degree 2. We can explain the last equality as follows:

$H^*(\mathbb{S}^2)$ is isomorphic to \mathbb{C} in degrees two and zero, and trivial in other degrees.

These two vector spaces distribute over the tensor product. This determines the module structure of $H_G^*(\mathbb{S}^2)$ over $H^*(BT)$, and its grading as a vector space over \mathbb{C} .

Chapter 3

Baird's Conjecture Concerning the Generalized Atiyah-Bredon Sequence

3.1 Atiyah-Bredon Sequence

Let X be a T -space, and let X_i be the union of all orbits of dimension less than or equal to i . That is, $X_i = \{p \in X : \dim(T_p) \geq n - i\}$, where $T_p = \{t \in T : t \cdot p = p\}$ is the stabilizer of p . By convention $X_{-1} = \emptyset$. These sets filter X , as $X^T = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n = X$. The action of T on X restricts to actions on each of the X_i , so the

spaces X_i are themselves T -spaces, and we can form their equivariant cohomology.

Theorem 3.1.1 (Atiyah-Bredon Sequence). *Let T be a torus of rank r , and X be a compact, finite G -CW-complex that is equivariantly formal. Then the following sequence of $H_T^*(*)$ modules is exact:*

$$0 \rightarrow H_T^*(X) \xrightarrow{i^*} H_T^*(X_0) \xrightarrow{\delta_0} H_T^{*+1}(X_1, X_0) \xrightarrow{\delta_1} H_T^{*+2}(X_2, X_1) \xrightarrow{\delta_2} \\ H_T^{*+3}(X_3, X_2) \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_r} H_T^{*+r}(X_r, X_{r-1}) \rightarrow 0,$$

where δ_i is the transition map from the exact sequence of the triple (X_{i-1}, X_i, X_{i+1}) .

Proof. [7] □

Note that because $X_{-1} = \emptyset$, δ_0 is just the transition map for the long exact sequence of the pair (X_1, X_0) .

Example 3.1.1. Consider $T = \mathbb{S}^1$ acting on $X = \mathbb{S}^2$ by rotation, as in Example 1.2.1. In Example 2.5.3, we determined the module structure of $H_T^*(\mathbb{S}^2)$. Using the Atiyah-Bredon sequence we can also determine its structure as a graded ring. The fixed point set $X_0 = X^T$ consists of two disjoint points, the north and south poles. The remaining orbits are one-dimensional, so that $X_{i>1} = X_1$. Since $X_2 = X_1$, only the first three terms of the Atiyah-Bredon sequence are non-trivial:

$$0 \rightarrow H_T^*(\mathbb{S}^2) \rightarrow H_T^*(X^T) \xrightarrow{\delta} H_T^{*+1}(\mathbb{S}^2, X^T) \rightarrow 0$$

Therefore the transition map δ is surjective in every degree.

Because T acts trivially on the fixed point set, by Example 2.5.1,

$$\begin{aligned}
 H_T^*(X^T) &= H^*(BT) \otimes_{\mathbb{C}} H^*(X^T) \\
 &= H^*(\mathbb{C}P^\infty) \otimes_{\mathbb{C}} (\mathbb{C} \oplus \mathbb{C}) \\
 &= H^*(\mathbb{C}P^\infty) \oplus H^*(\mathbb{C}P^\infty) \\
 &= \mathbb{C}[u] \oplus \mathbb{C}[u].
 \end{aligned}$$

Meanwhile, by Theorem 2.3.1,

$$\begin{aligned}
 H_T^*(\mathbb{S}^2, X^T) &= H^*(\mathbb{S}^2/T, X^T/T) \\
 &= H^*(\mathbb{I}, X^T) \\
 &= \mathbb{C}.
 \end{aligned}$$

The reader is encouraged to work out the last equality by considering the long exact sequence of the pair (\mathbb{I}, X^T) .

Because $H_T^*(\mathbb{S}^2, X^T)$ is trivial except in degree one, the above exact sequence gives us an isomorphism $H_T^k(\mathbb{S}^2) \cong H_T^k(X^T)$ for $k > 1$. We noted earlier that equivariant cohomology reduces to regular cohomology in degree 0. Hence $H_T^0(\mathbb{S}^2) = \mathbb{C}$, and we have

$$H_T^*(\mathbb{S}^2) = \begin{cases} \mathbb{C} & \text{in degree 0} \\ \mathbb{C}[u] \oplus \mathbb{C}[u] & \text{in degree } 2n \end{cases}$$

As a graded ring it is a subring of $\mathbb{C}[u] \oplus \mathbb{C}[u]$ generated by $1 \oplus 1$, $u \oplus 0$, and $0 \oplus u$.

It can be shown that this ring is isomorphic to $\mathbb{C}[x, y]/(x^2 - y^2)$ through the map sending $x \mapsto u \oplus u$ and $y \mapsto u \oplus -u$.

3.2 Generalized Atiyah-Bredon Sequence

In the hypotheses of Theorem 3.1.1, we required that X be a T -space. An obvious question to ask is whether the Atiyah-Bredon sequence continues to hold for all compact connected Lie groups. The answer is false, which can be seen from the next example.

Example 3.2.1. Let $X = G = SU(2)$, the group of 2×2 unitary matrices with determinant 1. It can be shown that when G acts on X by conjugation, X is an equivariantly formal G -space. This action has two fixed points, $\pm I$, where I is the identity matrix. Thus, in the above notation, $X_0 = \{I, -I\}$, a discrete two point set, and we have $H_G^0(X_0) \cong H^0(X_0) \cong \mathbb{C}^2$. To find $H_G^0(X)$, consider that both EG and X are connected, whereby $EG \times X$ is also connected. Then since $EG \times_G X$ is the continuous image of a quotient map from $EG \times X$, it must also be connected as well. Hence $H_G^0(X) = H^0(EG \times_G X) = \mathbb{C}$.

As a Lie group, $SU(2)$ has the Lie algebra $\mathfrak{su}(2) = \mathbb{R}^3$, where multiplication of elements corresponds to the cross product. Since the cross product of two linearly

independent vectors x and y is a third vector orthogonal to both x and y , \mathbb{R}^3 cannot have a two-dimensional subspace closed under the cross product. Therefore $\mathfrak{su}(2)$ has no two-dimensional Lie subalgebra. This in turn means that $SU(2)$ has no two-dimensional Lie subgroup, because the Lie algebra of such a subgroup would be a two-dimensional subalgebra of $\mathfrak{su}(2)$.

Since $SU(2)$ has no two-dimensional Lie subgroups, in particular it has no two-dimensional stabilizer subgroups. As a result, it has no one-dimensional orbits, and $X_1 = X_0$. This implies that the relative cohomology groups $H_G^*(X_1, X_0)$ are trivial, and that the first four terms of the Atiyah-Bredon sequence reduce to

$$0 \rightarrow H_G^*(X) \xrightarrow{i^*} H_G^*(X_0) \rightarrow 0.$$

For this sequence to be exact, i^* would have to be an isomorphism. But we showed above that $H_G^0(X) = \mathbb{C}$, while $H_G^0(X_0) = \mathbb{C}^2$. This proves that Theorem 3.1.1 does not apply for $G = SU(2)$.

Let G be a compact, connected Lie group, and X be a G -space. By Theorem 1.1.1, G contains a maximal torus, and we define the *rank* of G to be the dimension of its maximal torus. Let $G_p = \{g \in G : g \cdot p = p\}$ be the stabilizer of p . It is straightforward to show that G_p is a compact Lie subgroup. Therefore the identity component G_p^0 of G_p is a compact, connected Lie group and thus has a maximum

torus.

To construct the Atiyah-Bredon sequence in Theorem 3.1.1 we filtered X by considering the dimensions of the stabilizers. Baird has conjectured that the Atiyah-Bredon sequence does continue to hold for non-abelian compact Lie groups if we consider instead the ranks of the stabilizers.

Conjecture 3.2.1 (Generalized Atiyah-Bredon Sequence). *Let G be compact, connected, rank r Lie group, and X a compact finite G -CW-complex that is equivariantly formal. Let $X_i = \{p \in X : G_p^0 \text{ has rank } \geq r - i\}$. Then the following sequence for the filtration $X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r$ is exact.*

$$0 \rightarrow H_G^*(X) \xrightarrow{i^*} H_G^*(X_0) \xrightarrow{\delta_0} H_G^{*+1}(X_1, X_0) \xrightarrow{\delta_1} H_G^{*+2}(X_2, X_1) \xrightarrow{\delta_2} \\ H_G^{*+3}(X_3, X_2) \xrightarrow{\delta_3} \cdots \xrightarrow{\delta_r} H_G^{*+r}(X_r, X_{r-1}) \rightarrow 0,$$

where δ_i is the transition map from the long exact sequence of the triple (X_{i-1}, X_i, X_{i+1}) .

Exactness of the first four terms has already been established by Goertsches, Oliver and Mare, a result known as the non-abelian Chang-Skjelbred Lemma [9].

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